MINIMUM FUEL ORBITAL TRANSFERS

(Coaxial, Coplanar Elliptic Orbits, Unlimited Transfer Time.)

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United Aircraft Research Laboratories

from La Recherche Aerospatiale, No. 105, March-April 1965.

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SUMMARY

The following study constitutes one of the rare examples for which the application of Pontryagin's "Maximum Principle" to an orbital transfer problem permits attainment of the optimal solution in a closed form.

The elements of the Keplerian osculating orbit have been chosen as state co-ordinates because these elements remain constant on the ballistic arcs. The equations of motion during powered flight then coincide with the perturbation formulas concerning the elements of the Keplerian osculating orbit.

When the calculation gives several solutions at the same time, it is necessary to make a direct computation to determine the best one and to eliminate the unwanted ones, because the "Maximum Principle" is only a necessary condition for optimality in the case of a non-linear system.

PRINCIPAL NOTATIONS

A = apogee

c_i = constants

$\vec{F}$ = thrust

F = $|\vec{F}|$

m = mass of the moving body

$\vec{p}$ = adjoint vector

P = perigee

V = velocity of the moving body

V_{car} = characteristic velocity

V^*_{car} = reduced characteristic velocity

v = true anomaly

$\vec{x}$ = "state" vector

$\vec{y}$ = "control" vector

ϖ = angle between the tangent and the local horizontal

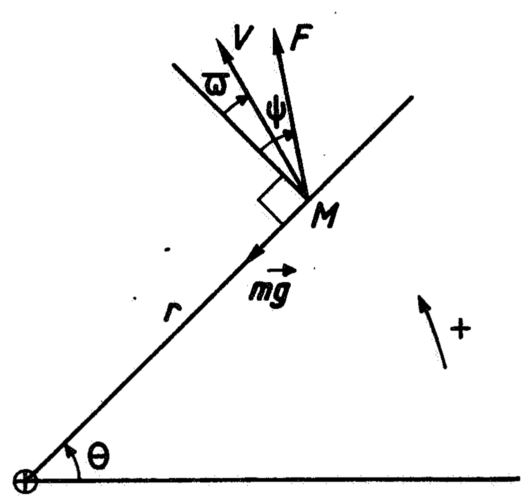
ψ = angle of the thrust with the local horizontal

ϕ = characteristic velocity measured from the initial time

I. INTRODUCTION

The study will be limited to the following problem: One wishes to accomplish a minimum fuel transfer (minimum of propellants without regard to the duration) in the gravity field created by a single center of attraction (Fig. 1). The transfer is between 2 Keplerian orbits (O) and (F) which are coplanar ellipses in which the axes' directions are not assigned.

Fig. 1.



II. FORMULATION OF PONTRYAGIN

One adopts Pontryagin's formulation such as it was shown, for example, in Refs. 1, 2, and 3.

II.1. STATE

At each moment (t) the state of the moving body M will be characterized by the state vector:

$$\vec{x} \left| \begin{array}{l} x_1 = h^2 \\ x_2 = E \\ x_3 = \phi \end{array} \right.$$

where $E = \frac{1}{2}V^2 - \mu/r$ and $h = rV \cos \varpi$

are respectively, the energy and the angular momentum of the Keplerian orbit, which is the osculating ellipse* at the time (t), and

$$\phi = \int_0^t \frac{F}{m} dt = -\int_{m_0}^{m} W(m) \frac{dm}{m} \geq 0$$

is the characteristic velocity "consumed from the time of departure."

E and h perfectly define the orbital state $\vec{x}_b$ of the moving body M. They are sufficient to determine the "osculating" orbit. (Neither the position on this orbit nor the orientation of the axis have to be taken into account.)

The speed of ejection W(m) was supposed to be a known function of instantaneous mass of the moving body. Since ϕ is representative of the consumed mass, it can serve as a measure of the expense of the maneuver up to the time (t), independent of staging considerations (Ref. 4).

II.2. CONTROL

The control vector $\vec{y}$ is made up of the components:

$$\vec{y} \left| \begin{array}{l} y_1 = \psi \\ y_2 = v \\ y_3 = F/F_{max}(m) \end{array} \right.$$

In effect, the choice of the direction of thrust is arbitrary. The choice of the point on the osculating orbit where the thrust is applied is equally arbitrary because the duration of the transfer does not enter into the calculation. One can wait on the osculating orbit until the return to optimal conditions for the thrust to be applied again.

On the other hand, it is supposed that the magnitude of thrust is variable, but limited by the following constraint: $0 \leq y_3 \leq 1$.

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* That is to say, the Keplerian orbit which would be described by M if, from that instant one suppresses the thrust $\vec{F}$. It is easy to show that it ought to be found in the plane of (O) and (F).

II.3. EQUATIONS OF MOTION

The equations concerning the "geometry" of the transfer are analogous to the formulas of perturbations of the elements in a Keplerian orbit.

$$(3) \quad \frac{dh}{dt} = r \frac{F}{m} \cos \psi$$

$$(4) \quad \frac{dE}{dt} = V \frac{F}{m} \cos (\psi - \varpi)$$

The equation concerning the fuel consumption is written:

$$(5) \quad \frac{d\phi}{dt} = \frac{F}{m}$$

These equations remain verified if one multiplies the linear dimensions by the scale factor λ, and the time by the factor $\lambda^{3/2}$. Then the velocities (including the characteristic velocity ϕ) are multiplied by $\lambda^{-1/2}$, E by λ^{-1}, and h by $\lambda^{1/2}$, which preserves the shape parameter e (eccentricity of the osculating orbit) which only depends on the product Eh^2.

Expenditures of characteristic velocity in the inverse proportion of the square root of the scale factor correspond to similar transfers. The study of such transfers can be reduced to the study of any one among them.

The duration of the transfer was not assigned. Let us take ϕ as a variable in place of t. The equations (3), (4) and (5) are then written:

$$(6) \quad x_1' = \frac{2\sqrt{x_1}\, p \cos y_1}{1 + e \cos y_2} = f_1(\vec{x}, \vec{y})$$

$$(7) \quad x_2' = \frac{\sqrt{x_1}}{p}\left[\cos y_1 + e \cos (y_1 + y_2)\right] = f_2(\vec{x}, \vec{y})$$

$$(8) \quad x_3' = 1 = f_3(\vec{x}, \vec{y}) \quad \text{with } (\ )' = \frac{d}{d\phi}$$

where p (semi-latus rectum of the osculating orbit) and e depend on x_1 and x_2.

$$(9) \quad p = \frac{h^2}{\mu} = \frac{x_1}{\mu}$$

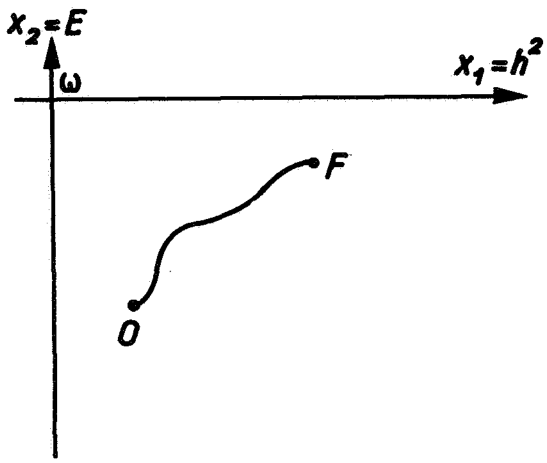
$$(10) \quad e = \left(1 + \frac{2Eh^2}{\mu^2}\right)^{1/2} = \left(1 + \frac{2x_1 x_2}{\mu^2}\right)^{1/2}$$

The problem consists then of choosing in which manner the control y should be applied to pass from the given initial orbital state, x_b^o , to the final given orbital state, x_b^f , in the most economical manner. That is to say, such that:

$$(11) \quad S = \sum_{i=1}^{3} c_i x_i(\phi_f) = x_3(\phi_f) = \phi_f$$

is minimal (Fig. 2).

Fig. 2. Trajectory in the plane $x_1 x_2$



ϕ_f is the characteristic velocity of the mission, V_{car}.

The problem is of the "minimum time" type, Ref. 2, the role of the time being played here by ϕ .

II.4. ADJOINT VECTOR

Let us define an adjoint vector $\vec{p}$ of which the components p_i must satisfy the equations:

$$(12) \quad p_i' = -\sum_{s=1}^{3} p_s \frac{\partial f_s}{\partial x_i} \quad (i = 1, 2, 3)$$

the f_s being the right hand sides of the equations (6), (7) and (8).

II.5. MAXIMUM PRINCIPLE

From the above considerations the maximum principle of Pontryagin is expressed as follows: The "optimal trajectory" is obtained by integrating the differential system: (6) and (7), and (8) and (12), with unknowns $\vec{x}$ and $\vec{p}$, in which the control vector, $\vec{y}$, has been chosen such that at each instant the generalized Hamiltonian:

$$(13) \quad H = \vec{p}\cdot\vec{x}' = \sum_{i=1}^{3} p_i f_i = \sqrt{x_1}\left[p_1 \frac{2p\cos y_1}{1+e\cos y_2} + \frac{p_2}{p}\left(\cos y_1 + e\cos(y_1+y_2)\right)\right] + p_3$$

is an absolute maximum with respect to y_1, y_2, y_3.

The boundary conditions are:

$$(14) \quad \left|\begin{array}{l} x_1(0) = x_1^0 \\ x_2(0) = x_2^0 \\ x_3(0) = 0 \end{array}\right. \qquad\qquad (15) \quad \left|\begin{array}{l} x_1(\phi_f) = x_1^f \\ x_2(\phi_f) = x_2^f \\ p_3(\phi_f) = -c_3 = -1 \end{array}\right.$$

where x_1^0, x_2^0, x_1^f, x_2^f are given.

The supplementary equation: $H(\phi_f) = 0$ then determines ϕ_f.

III. OPTIMAL CONTROL

III.1.

H does not contain y_3. It is maximal with respect to y_1 and y_2 for $y_1 = 0$ or π, $y_2 = 0$ or π. That is:

$$\cos y_1 = \epsilon_1 = \pm 1, \quad \cos y_2 = \epsilon_2 = \pm 1$$

That signifies that the thrust is applied only at the perigee (P) or at the apogee (A) of the osculating orbit, in the direction of the velocity or in the opposite direction. The ambiguity in sign is removed by the absolute maximum condition on H with respect to ϵ_1 and ϵ_2.

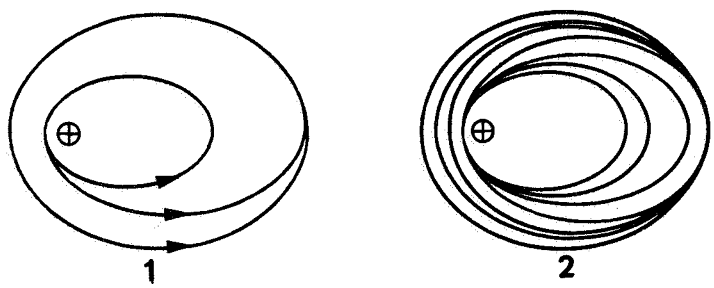
An immediate consequence of this result is that <u>the application of the thrust</u>

has to be discontinuous and impulsive*. After having applied the thrust at the perigee, for example, it is necessary to coast, in purely ballistic flight on the osculating orbit, to the perigee (or apogee) in order to apply the thrust again. Moreover, this thrust can only be applied during an infinitely short interval dt, in order not to depart from the conditions of the optimum.

It should be noted that we do not obtain any information about y_3, that is on the magnitude F of $\vec{F}$. This is due to the fact that the duration of the transfer does not enter the equation. Every value $F \leq F_{max}$ is suitable during dt.

If F_{max} is not infinite, the transfer takes place in an infinite time with an infinite number of infinitely small impulses, $dI = Fdt$. This is also true in taking $F = F_{max}$ (Fig. 3.2). The trace of the motion then appears continuous in the plane x_1, x_2 (Fig. 2). If F_{max} is infinite, one can perform the transfer in a finite time by finite impulses, $I = F_{max}dt$ (Fig. 3.1). The trace of the motion is then described by jumps in the x_1, x_2 plane.

Fig. 3.--1. $F_{max} = \infty$. Finite duration
2. F_{max} limited. Infinity of impulses, infinite duration.



In practice, F_{max} is not infinite and it is necessary to perform the transfer in a finite time. This is accomplished by adopting a solution neighboring the above-defined optimum, that is, to apply the thrust during a finite interval, Δt. With Δt fixed, the number of impulses (consequently the duration of the transfer) will then be rendered minimal if one chooses $F = F_{max}$.

Further on, when we speak of transfer by one, two or three impulses, we will assume implicity that F_{max} is infinite. Thus, if it is not, it will be necessary to replace each impulse by an infinity of infinitely small impulses applied at the proper point.

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* We will see in § VI.3 that the spiral trajectory solution with infinitesimal impulses applied over an infinite number of revolutions, such that the osculating orbit is always a circle ($y_1 \equiv y_2 \equiv 0$), is not locally optimal.

III.2. CHOICE OF ϵ_1 AND ϵ_2.

Inserting the values $y_1 = 0$ or π, $y_2 = 0$ or π in (13) the Hamiltonian becomes:

$$H = \sqrt{x_1}\,\Theta_2\epsilon_1 + p_3 \tag{16}$$

with $$\Theta_2 = \frac{2p_1 p}{1 + e\epsilon_2} + \frac{p_2}{p}(1 + e\epsilon_2).$$

We must choose ϵ_1 and ϵ_2 in such a way that H is an absolute maximum at each instant. As ϵ_1 can be chosen equal to ± 1, $|\Theta_2|$ must be rendered a maximum with respect to ϵ_2.

Let us introduce the reduced variables:

$$(17)\quad \left|\begin{array}{ll} X_1 = \dfrac{x_1}{\mu r} = \dfrac{p}{r} = 1 + e\epsilon_2 & \\ X_2 = \dfrac{x_2}{\left(-\dfrac{\mu}{2r}\right)} & (> 0 \text{ for an ellipse}) \\ Y = 2r^2\dfrac{p_1}{p_2} & (\text{accounts for the orientation of the vector } \vec{p}) \end{array}\right.$$

In these equations, $r = p/(1 + e\epsilon_2) = r_P$ (or r_A) = radius of the perigee <u>or</u> of the apogee of the osculating orbit. (The apsis radius, r, stays constant during a phase where ϵ_2 keeps a constant sign: for example, after an impulse at the perigee P, the same point P would be passed after a revolution.)

Then $\epsilon_2 = \pm 1$ depending on whether $|\Theta_2|_{\epsilon_2=+1} \gtrless |\Theta_2|_{\epsilon_2=-1}$ that is to say:

$$\Theta_1 = \frac{4ep_2^2}{r^2X_2^2}\left(-Y^2 + \frac{X_2^2}{X_1^2}\right) \gtrless 0. \tag{18}$$

Θ_1 is a switching function. The thrusting point depends on its sign.

Once ϵ_2 is chosen, $\epsilon_1 = \pm$ depending on whether:

$$\Theta_2 = \frac{p_2}{r}(Y + 1) \gtrless 0. \tag{19}$$

Θ_2 is a second switching function. On its sign depends the direction in which thrust is applied.

To study the sign of Θ_1 and Θ_2 it is necessary to integrate the equations of motion in which $\vec{y}$ has been replaced by $\vec{y}_{opt}$.

IV. INTEGRATION OF THE EQUATIONS--OPTIMAL ARC

For a determined phase (interval during which ϵ_2 keeps a constant sign) the equations (6), (7), (8), and (12) are written:

$$(20) \qquad \left| \begin{array}{l} x'_1 = 2\sqrt{x_1}\, r\epsilon_1 \\ x'_2 = \dfrac{\sqrt{x_1}\,\epsilon_1}{r} \\ x'_3 = 1 \end{array} \right.$$

$$(21) \qquad \left| \begin{array}{l} -p'_1 = \dfrac{\epsilon_1}{2r\sqrt{x_1}}(p_2 + 2r^2 p_1) + \dfrac{\sqrt{x_1}\,\epsilon_1\epsilon_2}{2r^2 e\mu}(2r^2 p_1 - p_2) \\ -p'_2 = \dfrac{\sqrt{x_1}\,\epsilon_1\epsilon_2}{e\mu}(p_2 - 2r^2 p_1) \\ -p'_3 = 0 \end{array} \right.$$

where r is a constant.

These equations can be integrated to give the form of the optimal arc in the space (x,p).

IV.1.

In particular, the projection in the plane x_1, x_2 (or X_1, X_2) is, for the initial phase:

$$(22) \qquad x_2 - x_2^0 = \frac{x_1 - x_1^0}{2r^2} \text{ or } X_2 - X_2^0 = -(X_1 - X_1^0)$$

$$\text{or again} \qquad X_2 - 1 = -(X_1 - 1)$$

Equation (22) shows that the trajectory in the plane x_1, x_2 (or X_1, X_2) is a straight line, each of the ellipses having the same radius of perigee (or apogee) as (0).

Figures 4 and 5 show in detail the properties of the planes x_1 and x_2, and X_1, X_2. Having been given an ellipse (0), all the ellipses having the same radius of perigee as (0) are situated on the segment $\overset{\circ}{C_P}P_0$ tangent at C_P° to the hyperbola

$$\sqrt{1+\frac{2x_1x_2}{\mu^2}} \equiv e = 0.$$

Likewise, all the ellipses having the same radius of apogee as (0) are situated on the segment S_A° $\overset{\circ}{C_A}$, tangent to the same hyperbola at C_A° (5).

The optimal initial flight follows one of these segments.

Fig. 4. Properties of the plane E, h^2.

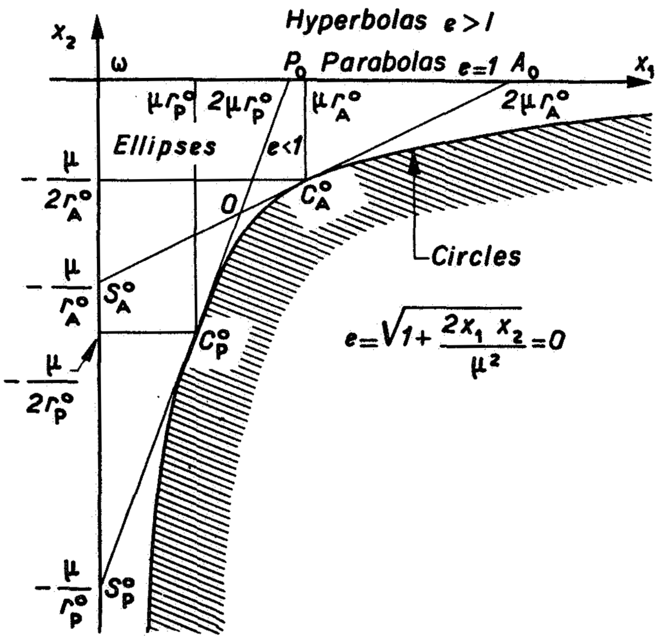
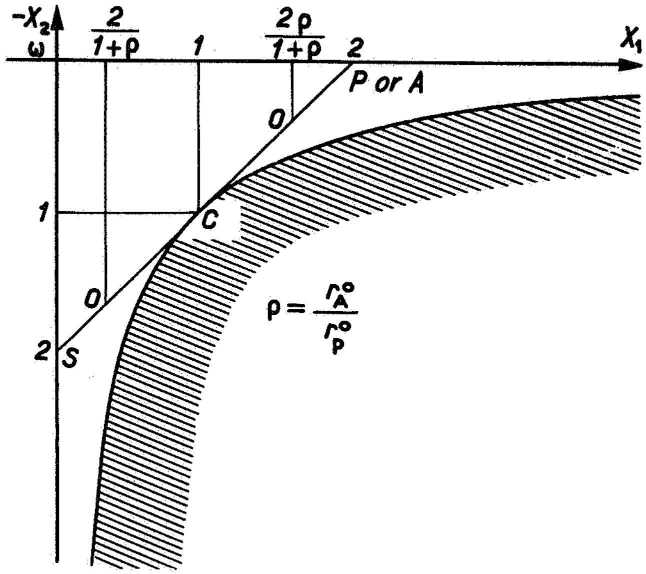


Fig. 5. Properties of the plane X_1, X_2.



IV.2.

Likewise the projection of the optimal arc in the plane X_1, Y is:

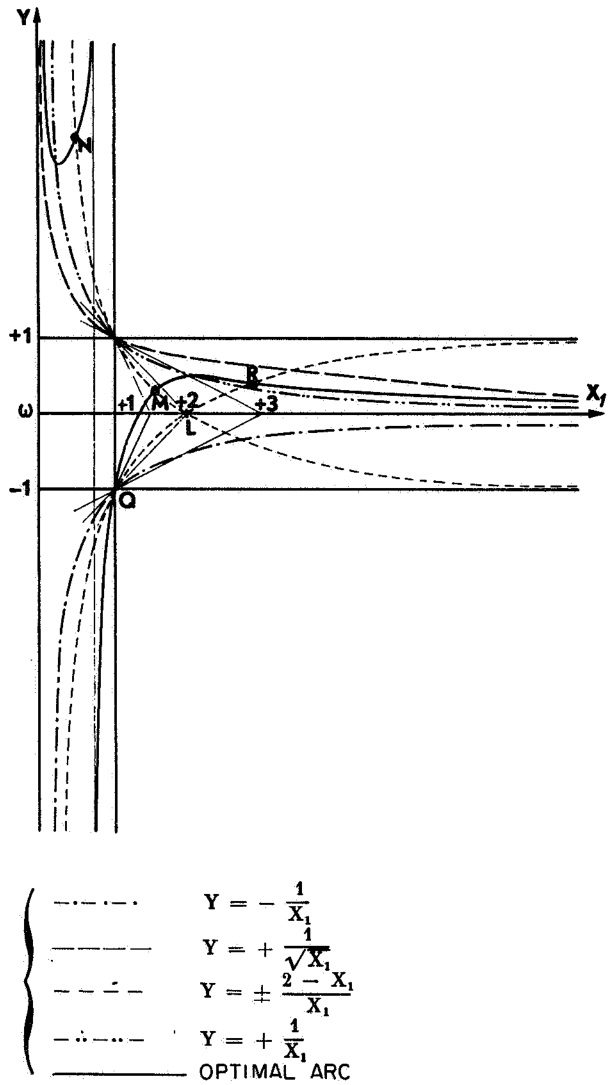
$$(23) \qquad \eta = \frac{1}{\sqrt{X_1^o}} \frac{1 + X_1^o Y_0}{1 + Y_0} = \frac{1}{\sqrt{X_1}} \frac{1 + X_1 Y}{1 + Y}$$

This equation is important, because it permits the study of the commutations. (Y enters in the switching functions, Θ_1, and Θ_2). The optimal arc in X_1, Y depends only on one parameter, η, which is a function of the initial conditions.

The fact that the two initial values, p_1^o and p_2^o (unknown a priori), are reduced to one single unknown parameter, Y^o (or η), can be accounted for by the homogeneous character of Equations (12) relative to p_1 and p_2. In effect, the coefficients of p_3 in these equations are zero because f_1 and f_2 do not contain ϕ. Moreover, since f_3 also does not contain ϕ, the system is conservative. Then, $H \equiv$ constant, and since $H(\phi_f) = 0$, $H \equiv 0$. Applying this relationship for $\phi = 0$, one obtains: $H(x_1^o, x_2^o, p_1^o, p_2^o) = 0$ which shows well that p_1^o and p_2^o are not independent.

The form of the optimal arc $Y = Y(X_1)$ is indicated in Fig. 6.

Fig. 6. The optimal arc and the points of commutation.



V. STUDY OF THE COMMUTATIONS (SWITCHES)

From Equations (18) and (22) one deduces that the frontier curves ($\Theta_1 = 0$) are expressed by:

$$Y = \epsilon \frac{2 - X_1}{X_1} \qquad \epsilon = \pm 1 \tag{25}$$

These curves are shown in Figs. 6, 7, and 8.

It is easy to demonstrate that the hatched zones of Fig. 8 are prohibited, and to eliminate the hyperbolic zone ($X_1 > 2$) for an economic transfer between two ellipses.

When, in the course of following an optimal arc, a point M, N or R (Fig. 6) is reached, there is a commutation. That is, ϵ_2 changes sign. The parameter r then changes character ($r_P \leftrightarrow r_A$) and it is necessary to set out from another point M, N or R situated on the curves of commutation. In doing this the continuity of the state variables x_1 and x_2, and of the adjoint variables p_1 and p_2 must be taken into consideration. It can be deduced that only the commutations $M \leftrightarrow N$ and $R \leftrightarrow R$ are allowed, thereby eliminating the commutations $M \not\leftrightarrow R$ and $N \not\leftrightarrow R$.

Fig. 7. Commutation curves

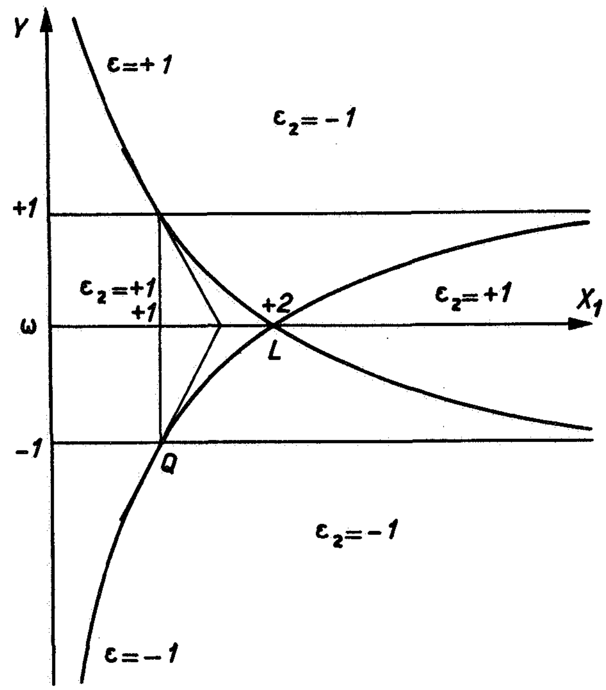
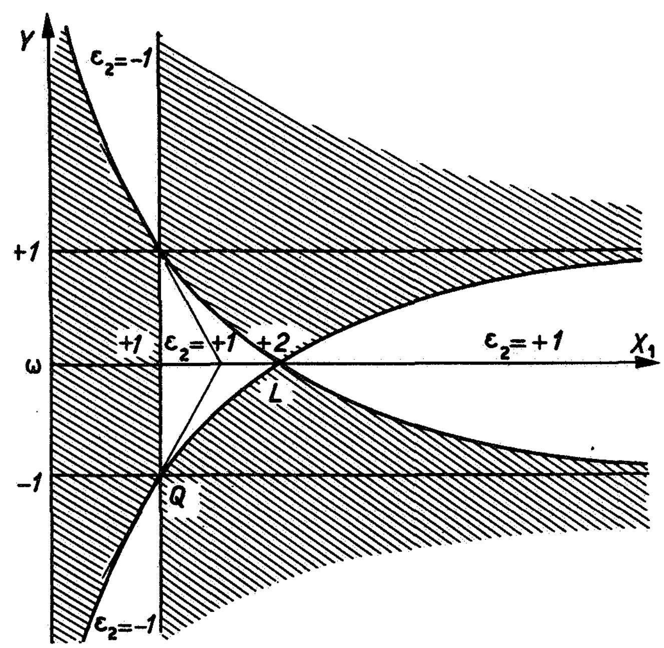


Fig. 8. Permitted zones



Crossing the point Q (circularization) is not optimal. It is possible to show by direct calculation that the path K_1MNK_2 is more economical than the path K_1CK_2 (Fig. 10).

When the optimal arc passes through L, there is a double commutation, beginning again from the same point L, but in the opposite direction.

The loci of the points M, N, R as functions of η are given in Fig. 9.

Fig. 9. Loci of the commutation points.

------------------	locus of M
— — — — — — — — — —	locus of N
— • — • — • — • — •	locus of R

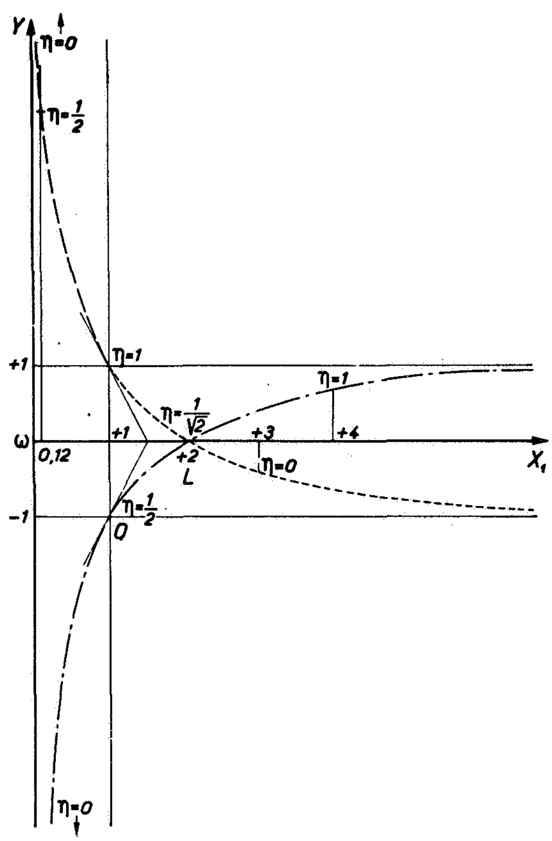
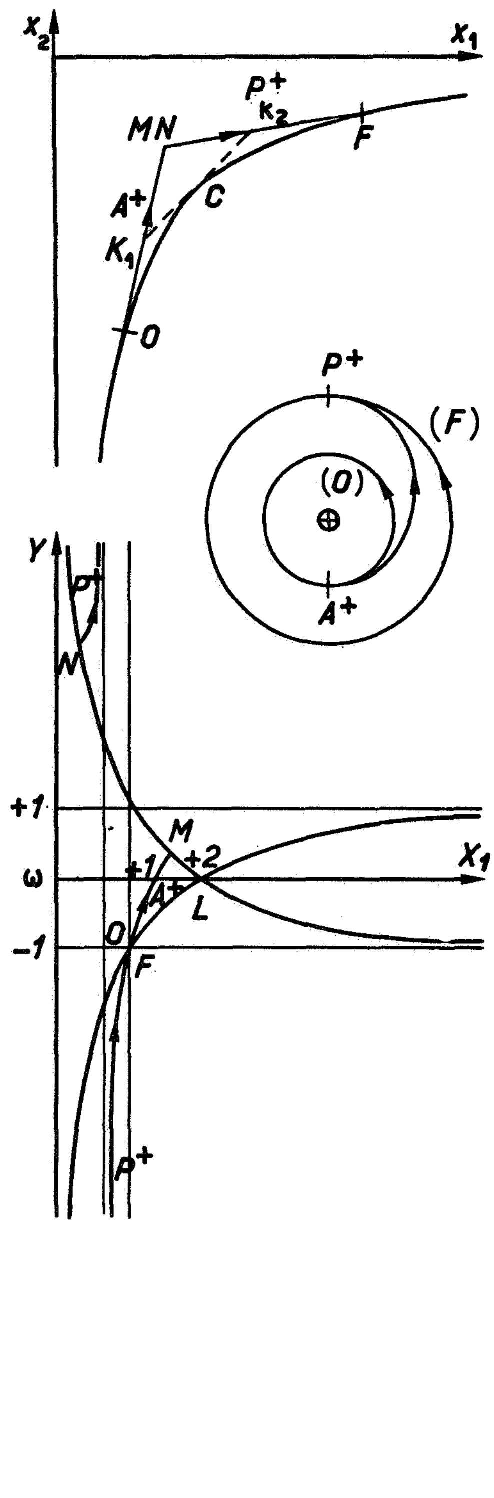


Fig. 10. Hohmann transfer (Case a).



VI. TRANSFERS BETWEEN CIRCULAR ORBITS

VI.1. APPLICATION OF THE MAXIMUM PRINCIPLE--EXTREMAL SOLUTIONS.

Leaving a circle (O) and following an optimal arc, one necessarily ends at a circle (F) after a commutation (Hohmann transfer (a) Fig. 10), 2 commutations (case (b) Fig. 11), or a double commutation (case (c) Fig. 12).

The solution (c) is always extremal.

Fig. 11. Transfer by 3 impulses (case b)

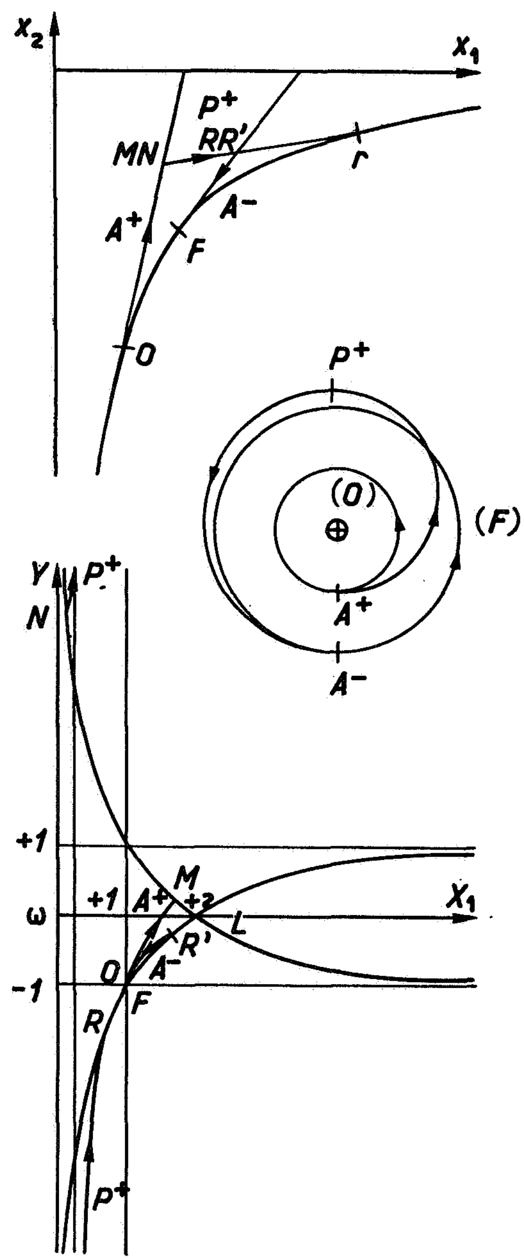
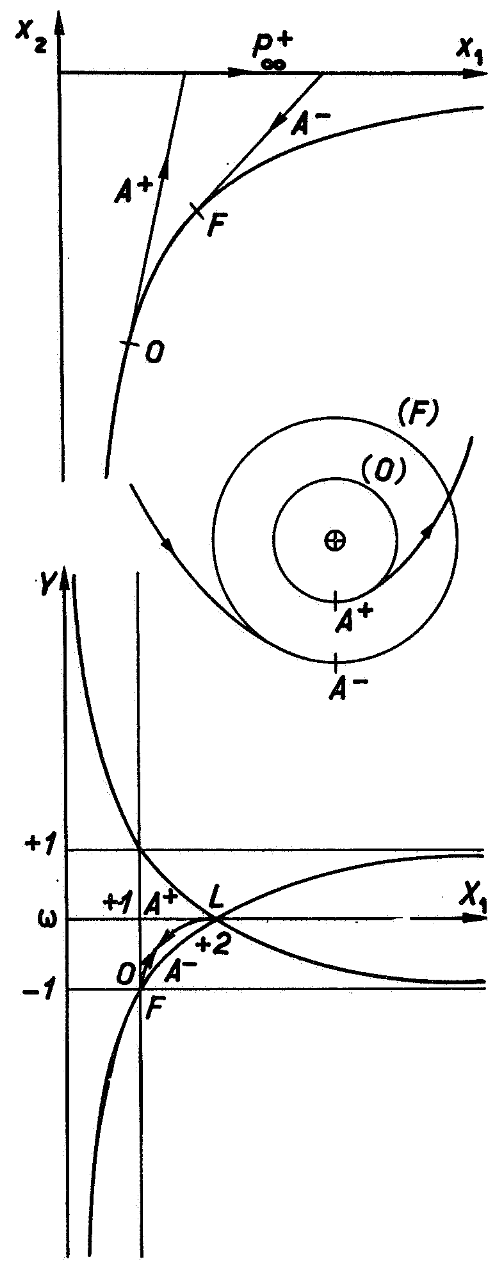


Fig. 12. Passage through infinity(bi-parabolic)(case c).



The solution (a) is only locally extremal if it is possible to join N to F without a new commutation of the type R (Fig. 11). That is, if:

$$0.12 < X_1^N < 1 \quad \text{(fig 9)}.$$

Since $X_1^N = 2/(1+\rho)$ (Fig. 5) with $\rho = r_f/r_o$, this condition is written

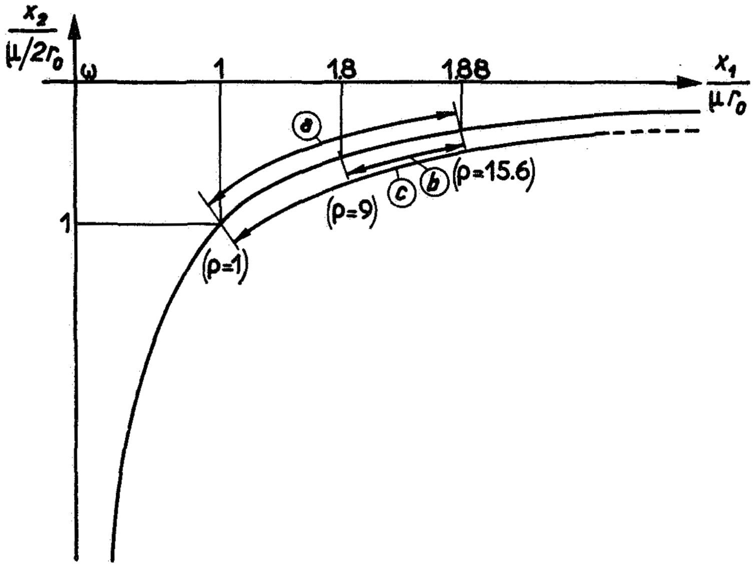
$$1 < \rho < 15.6$$

One could also show that the solution (b) is only extremal for $9 < \rho < 15.6$ and an exactly determined value of r.

These results are summarized in Fig. 13.

Fig. 13. Extremal solutions (Pontryagin).

a = Hohmann (2 impulses)
b = 3 impulses ⟶ local max.
c = 3 impulses of which one is at infinity



Note that for certain values of ρ, several solutions coexist. To determine the optimum and eliminate the other solutions, it is necessary to compare these solutions. A direct calculation is the only means of selecting the optimum.

VI.2. DIRECT CALCULATION

Putting $\rho_2 = r/r_f$ (where r is defined in Fig. 11), the characteristic velocities required for the different solutions, normalized by the circular orbital velocity on the final orbit, are:

$$\frac{\phi_f(b)}{\sqrt{\mu/r_f}} = \sqrt{2}(\rho\rho_2 - 1)\rho_2^{-1/2}(\rho\rho_2 + 1)^{-1/2} + \sqrt{2}\,\rho_2^{-1/2}(\rho_2 + 1)^{1/2} - \rho^{1/2} - 1$$

(c) corresponds to case $\rho_2 \longrightarrow \infty$ therefore:

$$\frac{\phi_f(c)}{\sqrt{\mu/r_f}} = (\sqrt{2} - 1)(1 + \rho^{1/2})$$

(a) corresponds to the case $\rho_2 \longrightarrow 1$ therefore:

$$\frac{\phi_f(a)}{\sqrt{\mu/r_f}} = \sqrt{2}\,(\rho - 1)(\rho + 1)^{-1/2} - \rho^{1/2} + 1$$

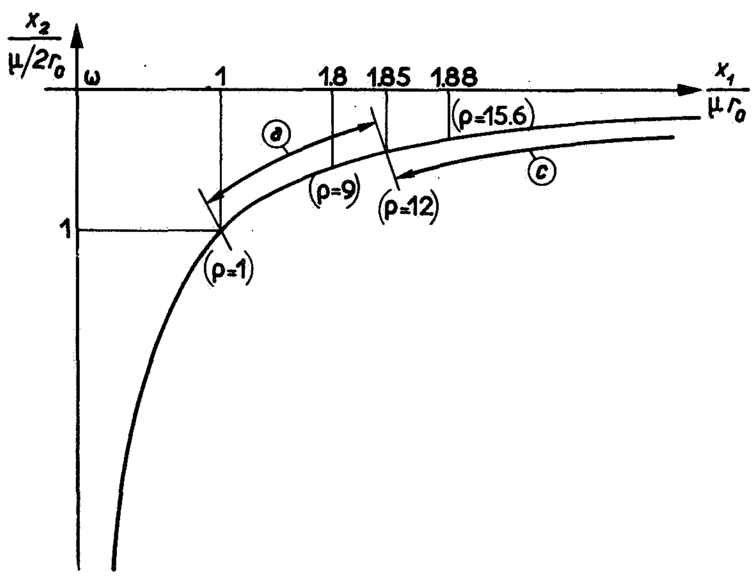
(Hohmann transfer)

The variations of required characteristic velocity as a function of ρ_2 for different values of ρ are shown in Fig. 15.

It is immediately evident that the solution (b) for $9 < \rho < 15.6$ and the solution (c) for $1 < \rho < 9$ are parasitic solutions because they correspond to local maxima. One finds them nevertheless, in applying the maximum principle, because this principle is only a necessary condition of optimality for non-linear systems.*

Comparisons of the solutions (a) and (c) show that: (a) is more economical than (c) for $1 < \rho < 11.94$; (a) is less economical than (c) for $\rho > 11.94$.

Fig. 14. "Optimorum" optimum (direct calculation).



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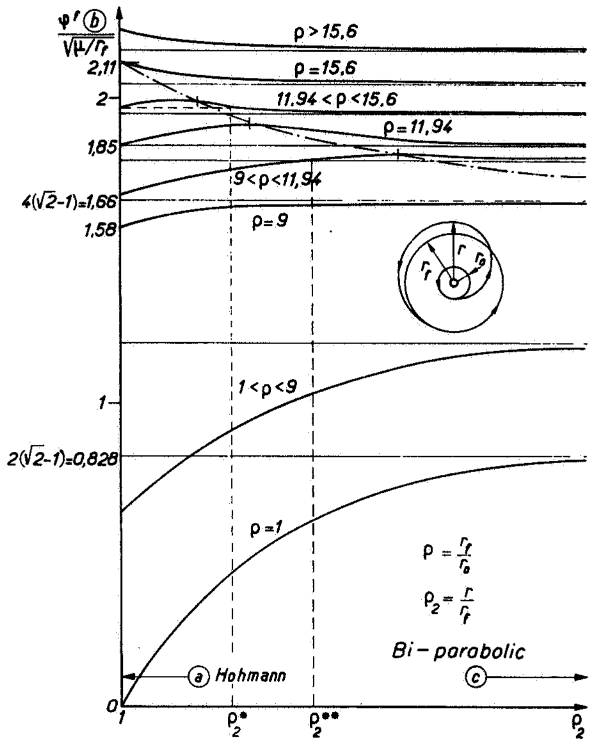
* A linear system would be of the form: $x'_i = f_i(\vec{x},\vec{y},\phi) = \sum_{s=1}^{3} a_{is}(\phi)\,x_s + b_i(\vec{y},\phi) \quad (i = 1,2,3)$

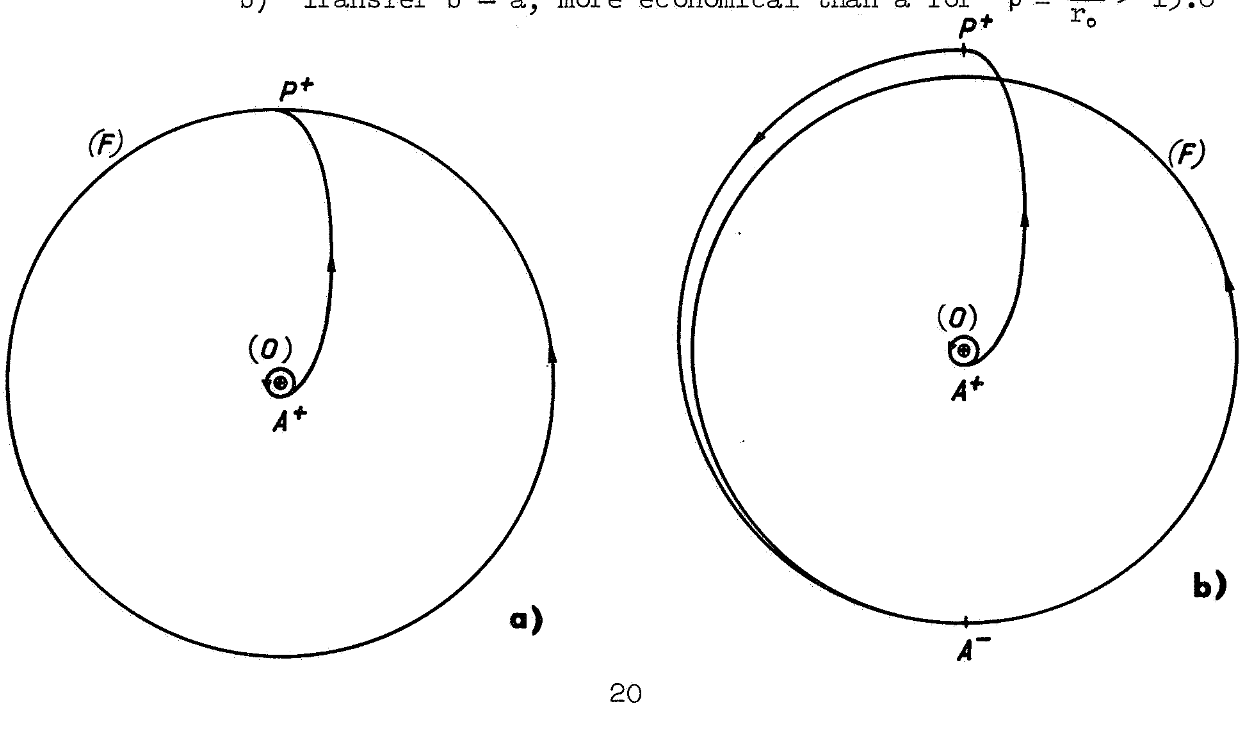
Fig. 15. Transfers of type b.



For ρ > 15.6 the solution (a) is not a local optimum (confirming the results deduced from the application of the Maximum Principle, Fig. 13).

One can find a three impulse transfer (b), infinitely close to the Hohmann transfer (a), which is more economical than the latter (Fig. 16).

Fig. 16. a) Transfer a, (Hohmann).
b) Transfer b ≃ a, more economical than a for $\rho = \frac{r_f}{r_o} > 15.6$



For $11.94 < \rho < 15.6$, every transfer by 3 impulses (b) corresponding to $\rho_2 > \rho_2^*$ (Fig. 15) is more economical than the Hohmann transfer (a), although not constituting a local maximum. However, it corresponds to an acceptable physical maneuver, when the solution (c) has no physical significance because it requires infinite duration.

The results of this discussion are in Fig. 14.

VI.3. SPIRAL

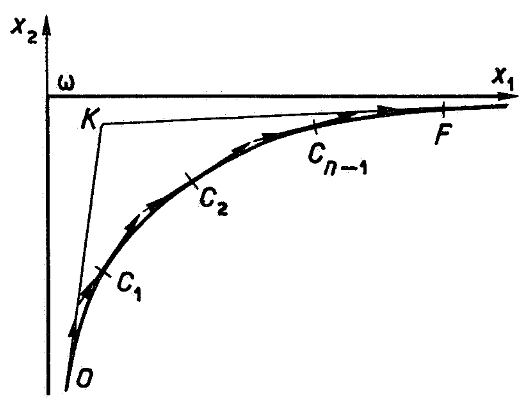
The path $\widehat{OF}$ (Fig. 18) on the hyperbola whose equation is:

$$\sqrt{1 + \frac{2x_1x_2}{\mu^2}} \equiv e = 0$$

in the plane x_1, x_2, corresponds to solution (d), a spiral with an infinite number of revolutions. This solution is not optimal. It is sufficient to consider it as the limit of the polygonal path $OC_1 \; C_2 \ldots\ldots C_n \equiv F$ (Fig. 17) when $n \longrightarrow \infty$ (chattering solution), which is not optimal because of the circularizations at $C_1 \; C_2 \ldots\ldots C_{n-1}$. One can demonstrate that the corresponding characteristic velocity required is:

$$\frac{\phi_f(d)}{\sqrt{\mu / r_f}} = \frac{V_0 - V_f}{V_f} = \rho^{1/2} - 1$$

Fig. 17. Polygonal contour $OC_1 C_2 \ldots\ldots F$ less economical than OKF.



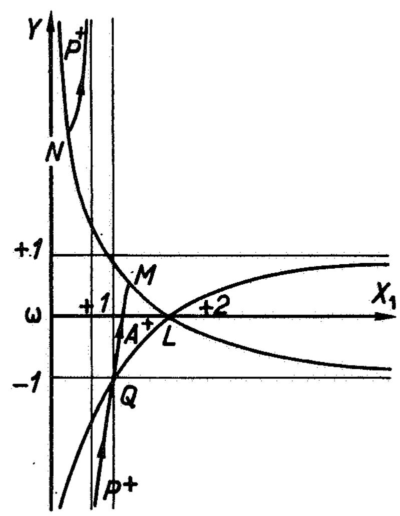
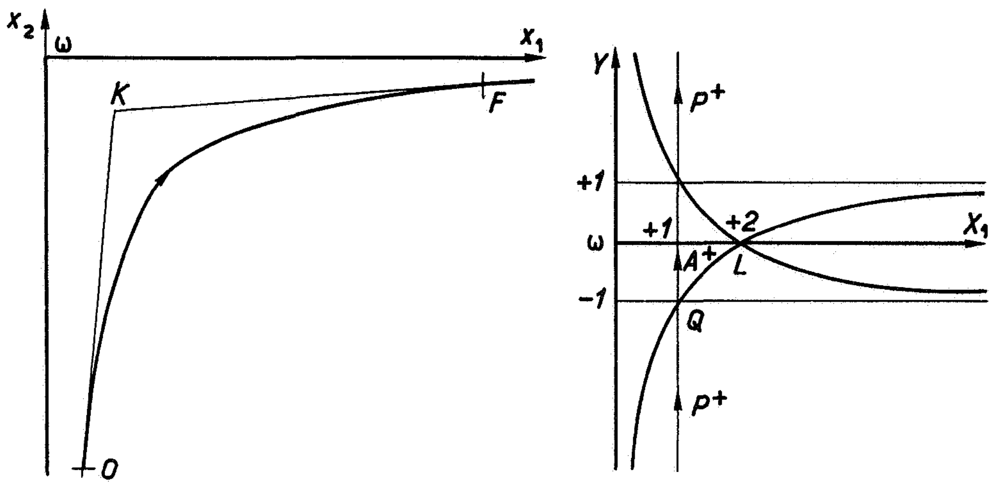


Fig. 18. Spiral ($n = \infty$ revolutions) $\widehat{OF}$ less economical than OKF.

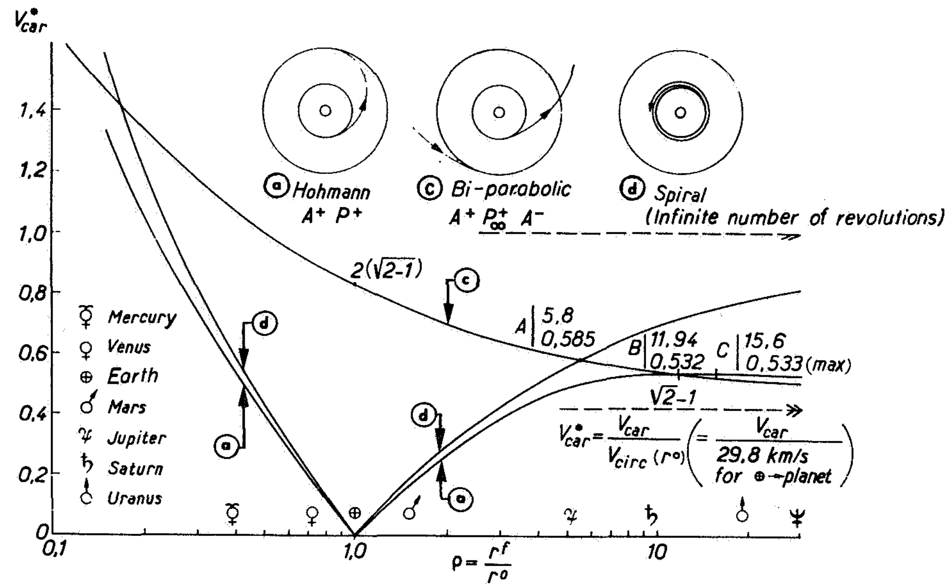


VI.4. CONCLUSION

We have traced, Fig. 19, the variations of characteristic velocity (or V_{car}) as a function of $\rho = r_f / r_o > 1$ in terms of the velocity on the circular orbit of departure $V_{circ}(r_o) = \sqrt{\mu/r_o}$ for the solutions: (a) Hohmann), (c) bi-parabolic, (d) spiral.

Although (c) is theoretically preferable to (a) for $\rho < 11.94$ the difference remains small. The Hohmann transfer is now, in general, the solution to adopt. Values of ρ greater than 11.94 are not often observed in practice. For circumterrestrial orbits we recall that, in order to transfer a satellite on a low orbit into a synchronous orbit, $\rho \simeq (42000 \text{ km}/6700 \text{ km}) \simeq 6.3 < 11.94$. Likewise, for interplanetary transfers toward the superior planets of the solar system, only Uranus, Neptune and Pluto are such that $\rho > 11.94$. For the latter planets, one cannot consider the solution (c) (infinite duration). The solution of the type (b) (with $\rho_2 > \rho_2^*$), although slightly better than the Hohmann transfer (a) (but less economical than solution (c)) presents the disadvantages of complexity (3 impulses in place of 2) and of increased duration, for a rather small savings compared to the Hohmann solution.

Fig. 19. Comparison of the modes of transfer between 2 circles (limited duration).



The spiral solution (d) is always less economical than the Hohmann. The ratio Φ_f (d)/ Φ_f(a) increases from 1 to $1/(\sqrt{2} - 1) = 2.41$ when ρ increases from 1 to infinity. Even for low thrust systems it is better to adopt the solution consisting of short, successive periods of thrust, at perigee and apogee of the osculating orbit (Fig. 3.2), rather than a solution of the spiral type, if the transfer duration is neglected.

The case where $\rho = r_f/r_o < 1$ (transfer toward the inferior planets, for example) is deduced simply from the case $\rho > 1$. If the characteristic speed is always related to circular speed on the departure orbit ($V_{circ}(r_o)$), it is sufficient to interpret $1/\rho > 1$ as the abscissa in Fig. 19, and to multiply the result read from the ordinate by $1/\sqrt{\rho} > 1$ (this is in accordance with the remarks of § II.3 concerning similar transfers).

One obtains then the curves ($\rho < 1$) of Fig. 12, which show particularly that transfers toward the inferior planets are more difficult to realize than the transfers toward the superior planets. For example, the Hohmann transfer toward Mercury ($\rho = 0.39$), for which the reduced characteristic speed is equal to 0.57, is already more difficult to realize than any Hohmann transfer toward a superior planet.

VII. TRANSFER BETWEEN ELLIPTIC ORBITS

Consider now a transfer between two ellipses (O) and (F) (Fig. 20). The only transfers permitted are, in view of the previous analysis, OKF (Fig. 21) and OK_1K_2F (Fig. 22) which are part of the optimal paths between circles: $C_P^o K C_A^f$ and $C_P^o K_1 K_2 C_P^f$.

Fig. 20.

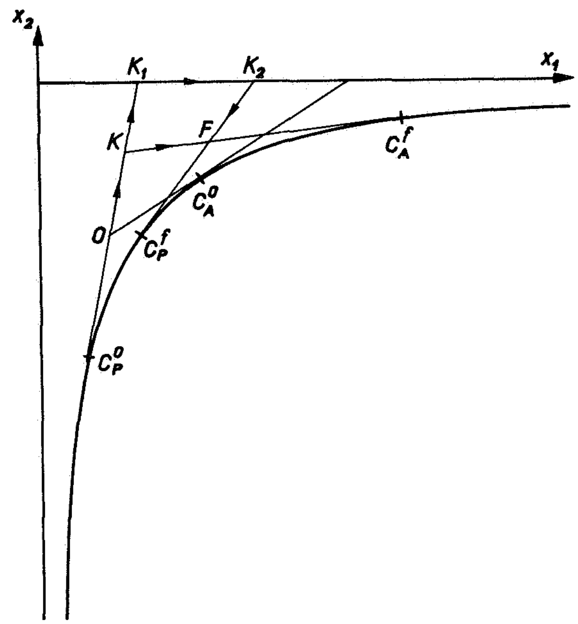


Fig. 21.

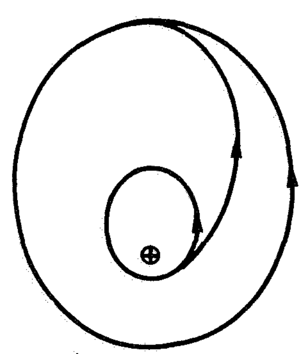
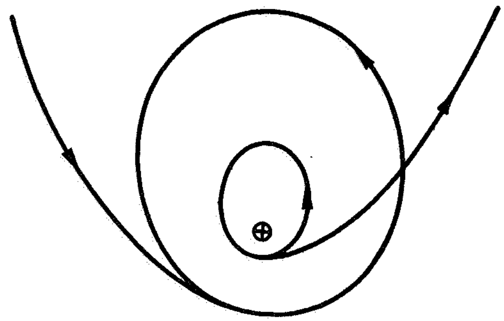


Fig. 22.



If $1 < r_P^f/r_P^o < 9$, the path $C_P^oKFC_P^f$ is surely more economical than the path $C_P^oK_1K_2C_P^f$. It is sufficient, in Fig. 15, to take $\rho = r_P^f/r_P^o$ and $\rho_2 = r_A^f/r_P^f$. Omitting from these paths the common parts C_P^oO and FC_P^f, one deduces that OKF is more economical than OK_1K_2F, whatever r_A^o and r_A^f are. If $(r_P^f/r_P^o) > 11.94$, analogous reasoning shows that OK_1K_2F is more economical than OKF, whatever r_A^o and r_A^f are. On the contrary, if $9 < (r_P^f/r_P^o) < 11.94$, OKF is more economical than OK_1K_2F only for $\rho_2 = (r_A^f/r_P^f) < \rho_2{}^{**}$ (Fig. 15). When ρ increases from 9 to 11.94 $\rho_2{}^{**}$ decreases from $\rho_2{}^{**} = \infty$ to $\rho_2{}^{**} = 1$.

In conclusion, studying the optima reveals that only the 2 perigee radii, r_P^o and r_P^f, enter into consideration when their ratio, $\rho > 1$, is between 1 and 9, or when ρ is greater than 11.94. When ρ is between 9 and 11.94 the greatest radius of apogee must be accounted for, as explained in the text. It is sufficient then to put ρ_2 = (greater radius of apogee/greater radius of perigee) and to consider its position with respect to $\rho_2{}^{**}$.

VIII. GENERAL CONCLUSION

Application of the Pontryagin Maximum Principle permits the selection of extremal solutions and, in particular, determination of the number* and points of application of the impulses.

A direct calculation is necessary in order to eliminate the parasitic solutions and to find the "optimal optimum."

In the case of transfers between circles, the Hohmann (bi-tangential, half-ellipse) is most economical when the ratio of the radii is less than 11.94. It is beneficial therefore, for most circumterrestrial and interplanetary

* Most of the studies of multi-impulse transfers suppose, a priori, a fixed number of impulses at one's disposal [6].

transfers. When the ratio of the radii is greater than 11.94, the solution consists of establishing a parabolic branch by the first impulse, returning on another parabolic branch after a second, infinitely small impulse applied at infinity, and finally, establishing the final circular orbit by a third impulse at the perigee. This transfer is theoretically more economical than the Hohmann (several %), but the duration is infinite.

In practice, it is possible to adopt a solution, not locally optimal, for which the second impulse is applied at a finite distance and not at infinity. But the economy realized in proportion to the Hohmann solution is very small.

In the solar system, the transfer from the terrestrial orbit toward the superior planets which necessitates the greatest expense is theoretically that which corresponds to a radius ratio of 11.94, that is, to an orbit situated between those of Saturn and Uranus. On the contrary, toward the inferior planets the expense always increases proportionately as the final orbit approaches the sun. The transfer toward an inferior planet is more difficult than the transfer toward a superior planet for the same radius ratio.

The case of transfer between two ellipses of the same major axis is easily deduced from the study of the transfers between circular orbits. Here again, it is necessary to choose between the Hohmann solution and the bi-parabolic solution. The latter can, in certain cases, represent a very important saving, in contrast to the small saving attainable in the circular case.

It is important to recall that all the preceding results suppose that the duration of the transfer does not come into play, and that these results will be profoundly modified when this is not the case.

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